(3) Particle moving near the surface of earth:-

Let z-axis be along upward vertical direction, then kinetic energy is

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

Further the applied force on the body is its weight acting in negative z-direction i.e.

$$F = F_z = -mg = -\frac{\partial V}{\partial z}$$

This gives $V = mgz$, on setting additive constant to zero. Now Lagrangian is

$$L = T - V$$

$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz,$$

so that, $\frac{\partial L}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = p_x = m\dot{x}$

giving, $\dot{x} = \frac{p_x}{m}$.

similarly, $p_y = m\dot{y}$ $\quad \dot{y} = \frac{p_y}{m}$

$p_z = m\dot{z}$ $\quad \dot{z} = \frac{p_z}{m}$

Hamiltonian for such a system is Conserved, i.e.

$$H = T + V$$
$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz$$
$$= \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + mgz$$

giving equations of motion

$$\left.\begin{aligned} \dot{p}_x &= -\frac{\partial H}{\partial x} = 0 \\ \dot{p}_y &= -\frac{\partial H}{\partial y} = 0 \\ \dot{p}_z &= -\frac{\partial H}{\partial z} = -mg \end{aligned}\right] \longleftarrow (1)$$

and

$$\left.\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \\ \dot{y} &= \frac{\partial H}{\partial p_y} = \frac{p_y}{m} \\ \dot{z} &= \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \end{aligned}\right\} \longleftarrow (2)$$

from eq. (2) on using eq. (1), we get

$$\left.\begin{aligned} \ddot{x} &= \frac{\dot{p}_x}{m} = 0 \\ \ddot{y} &= \frac{\dot{p}_y}{m} = 0 \\ \ddot{z} &= \frac{\dot{p}_z}{m} = -g \end{aligned}\right\} \longleftarrow (3)$$

which shows that acceleration along z-direction is the acceleration due to gravity and is true.

(4) Particle in a Central field of force:

In such a field, force is always directed towards the centre. Potential energy is the function of co-ordinates and the system is conservative so that Hamiltonian represents the total energy.

As the motion is always confined in a plane, we consider it in r, θ co-ordinates.

Kinetic energy is

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2)$$

and potential energy is

$$V = V(r)$$

So that Lagrangian is

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

Momenta are then

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \text{giving} \quad \dot{r} = \frac{p_r}{m}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \text{or} \quad \dot{\theta} = \frac{p_\theta}{mr^2}$$

Hamiltonian is

$$H = T + V$$

$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + V(r)$$

It follows therefore that

$$\left.\begin{aligned} \dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{\partial V}{\partial r} \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = 0 \end{aligned}\right\} \quad (1)$$

and,

$$\left.\begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \\ \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \end{aligned}\right\} \quad (2)$$

which are the desired equations of motion further from eq. (2), we can write

$$\ddot{r} = \frac{\dot{p}_r}{m} = \frac{p_\theta^2}{m^2 r^3} - \frac{1}{m} \cdot \frac{\partial V}{\partial r}$$

Or, $$m\ddot{r} = \frac{p_\theta^2}{mr^3} - \frac{\partial V}{\partial r} \quad \text{--- (3)}$$

We put $-\frac{\partial V}{\partial r} = F(r) =$ radial force

and $$\frac{p_\theta^2}{mr^3} = \frac{(mr^2\dot{\theta})^2}{mr^3} = mr\dot{\theta}^2 \quad \text{(using eq. 2)}$$

$$= \frac{m(r\dot{\theta})^2}{r}$$

$$= \frac{mv_\theta^2}{r} = \text{Centrifugal force}$$

Therefore, we have from eq. (3), that

$$m\ddot{r} = \frac{mv_\theta^2}{r} + F(r),$$

representing an equation of motion involving the actual force $F(r)$ and a centrifugal force $\frac{mv_\theta^2}{r}$. Since in central force motion whole of the force is along the radius vector, the latter is termed as fictitious force.